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A theory of mixing in a stably stratified fluid

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A theory is developed for turbulence in a stably stratified fluid, for example in the experiments of Rouse & Dodu and of Turner where there is no shear and the turbulence is induced by a source of energy near the lower boundary of the fluid. A growing mixed layer of thickness D appears in the lower portion of the fluid and is separated from the non-turbulent fluid above, in which the buoyancy gradient is given, by an interfacial layer of thickness h. The lower mixed layer has a very weak buoyancy gradient and the large buoyancy difference across the interfacial layer is Δb.

As indicated by the experiments of Thompson & Turner and Hopfinger & Toly, and derived by the author in a recent paper, if u is the root-mean-square horizontal velocity and l is the integral length scale, the eddy viscosity ul is a constant in a homogeneous fluid agitated by a grid. When there is stratification, the theory indicates that the fluid motion is unaffected by buoyancy forces in the mixed layer, so that ul should again be constant in the lower portions of the mixed layer. Since l is proportional to distance, we may conveniently suppose that the source of the disturbances is at a level $z = 0$ where u is infinite in accordance with $uz = K$. Thus we may take K to be a fundamental parameter characterizing the turbulent energy source. Then z is distance above the plane of the virtual energy source. If the non-turbulent fluid has uniform buoyancy, $D\Delta b = U^2$ may be shown to be constant. In general, whether constant or not, U may be taken to be a fundamental parameter expressing the stability. The quantity $\hat{Ri} = U^2D^2/K^2$ is the most fundamental of the several Richardson numbers that have been introduced in this problem because, with its use, 'constants' of proportionality do not depend on the molecular coefficients of viscosity or diffusion (for high Reynolds number turbulence) or on the geometry of the grid.

The theory contains a number of results:

(i) $u_e\, D/K \sim \hat{Ri}^{-\frac{7}{4}}$, where $u_e = dD/dt$ is the entrainment velocity. Integration yields $D \propto t^{\frac{2}{11}}$ for a homogeneous upper fluid and $D \propto t^{\frac{1}{9}}$ for a linear upper density field. This $-\frac{7}{4}$ entrainment law compares with a $-\frac{3}{2}$ law suggested by several experimenters.

(ii) Turbulence in the interfacial layer is intermittent with intermittency factor $I_3 \sim \hat{Ri}^{-\frac{3}{4}}$. The turbulent patches have dimension $\delta_3 \sim D\hat{Ri}^{-\frac{3}{4}}$.

(iii) If the (equal) root-mean-square velocities in an infinite homogeneous fluid at a distance D from the grid are denoted by $u_1 \sim v_1 \sim w_1$, we find that the r.m.s. velocities near the interface are $u_2 \sim u_1$, $v_2 \sim u_1$ and $w_2 \sim u_1\, \hat{Ri}^{-\frac{1}{4}}$.

(iv) The buoyancy flux q_2 near the interface may be expressed as $q_2 \sim w_2^3/D$.

(v) $h \sim D$, as observed in several experiments.

1. Introduction

Experimental observations beginning with those by Rouse & Dodu (1955) show that if a stably stratified fluid is agitated, say at the bottom of a container, a mixed layer develops near the bottom whose depth D increases with time. The observed mean buoyancy profile is shown schematically in figure 1. The turbulence dies out across an interfacial layer of thickness h across which there is a large buoyancy difference Δb. The problem has great importance in meteorology and oceanography as discussed by many authors, for example Kraus & Turner (1967).

Cromwell (1960) constructed a similar experiment to simulate the pycnocline but recent interest began with the careful measurements by Turner (1968). Subsequently many others have reported on identical or similar experiments (Brush 1970; Wolanski 1972; Linden 1973; Crapper & Linden 1974; Linden 1975; Wolanski & Brush 1975; Thompson & Turner 1975; Hopfinger & Toly 1976). Other experiments involving shear have been run by Kato & Phillips (1969), Moore & Long (1971), Wu (1973) and Kantha, Phillips & Azad (1977). Typical of these is that of Kato & Phillips, shown schematically in figure 2.

For the experiments without shear, experimenters have proposed that the entrainment velocity $u_e = dD/dt$ is given by†

$$u_e/fS \sim Ri^{*-\frac{3}{2}}, \quad Ri^* = D\Delta b/f^2S^2, \tag{1}$$

where Ri^* is the overall Richardson number, f is the frequency and S is the stroke of the grid. In shearing experiments there is some indication that $u_e/u_* \sim Ri^{*-1}$, where Ri^* is defined in terms of the friction velocity u_*, but considerable doubt has been raised by recent experiments (Kantha *et al.* 1977) in the Kato & Phillips tank.

A suggestion was made by Turner (1973) that the entrainment velocity should be expressed in terms of Δb and a velocity and length u_1 and l_1 characteristic of the r.m.s. velocity and the integral length scale evaluated in a region (region R_1) near the middle of the mixed layer. He suggested that $u_1 \sim fS$ and $l_1 \sim D$; then (1) implies

$$u_e/u_1 \sim Ri_1^{-\frac{3}{2}}, \tag{2}$$

where Ri_1 is the turbulent Richardson number defined in terms of Δb, u_1 and l_1. Long (1975), however, presented a uniform theory for all cases in which the buoyancy flux near the interfacial layer $q_2 \sim u_2^3/D$, where u_2 is the order of magnitude of the velocity components (all assumed of equal order) in the mixed layer near the interface (region R_2). This implies an Ri_2^{-1} law for u_e/u_2, where $Ri_2 = D\Delta b/u_2^2$. Long attempted to reconcile his theory and the expression in (2) by assuming that the small density variation in the mixed layer was sufficient to change the order of magnitude of the turbulent velocities from $u_1 \sim fS$ in the lower part of the layer to $u_2 \sim fSRi^{*-\frac{1}{6}}$, or $u_2 \propto f^{\frac{4}{3}}$. This conjecture was investigated experimentally by Hopfinger & Toly, who concluded that the velocities were everywhere proportional to f and that (2) is correct. The present paper finds that the turbulence near the interface becomes strongly anisotropic, in particular that $u_2 \sim v_2 \sim u_1$ (and are therefore proportional to f), but that

† In this paper, we assume strong stability in the sense that the overall Richardson number Ri^* or a similar non-dimensional number is large. If two non-dimensional numbers A and B have a ratio A/B tending to a finite, non-zero constant as $Ri^* \to \infty$, we say that A is of order B and write $A \sim B$. We use the proportionality symbol to connect two-dimensional quantities that vary together.

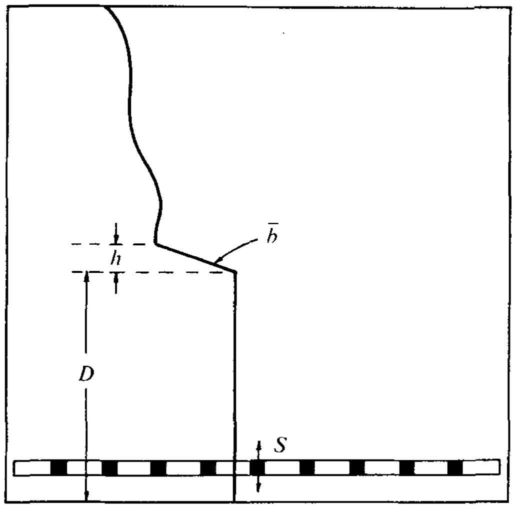

FIGURE 1. The container has a fluid with variable mean buoyancy profile $\bar{b}$. The lower layer of depth D is fully turbulent and has a weak buoyancy gradient. The buoyancy decreases strongly in an interfacial layer of thickness h between the mixed layer and the non-turbulent layer above, the buoyancy difference across h being Δb. The upper layer has a buoyancy gradient $b(z)$. The turbulence is caused by a grid oscillating up and down with stroke S and frequency f. The geometry and location of the grid are given by lengths $M_1, M_2, \ldots$.

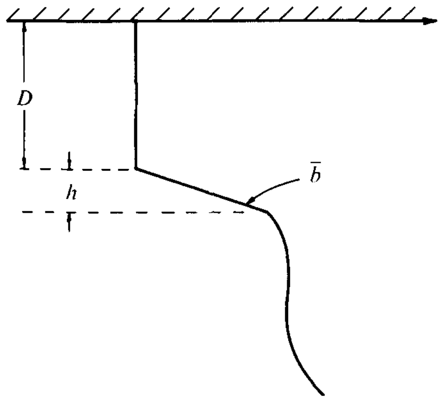

FIGURE 2. Kato & Phillips experiment with mean buoyancy profile. The turbulence in the upper layer is caused by a screen moving along the surface and exerting a stress $\tau = u_*^2$, where u_* is the friction velocity.

w_2 becomes small for high Richardson numbers. We now find $q_2 \sim w_2^3/D$ and since $w_2 \ll u_1$, we obtain a slower entrainment rate, namely $u_e/u_1 \sim Ri_1^{-\frac{7}{4}}$. The decrease in the turbulent velocity w_2 near the interface is explained by a process similar to that discussed by Hunt & Graham (1978) in connexion with homogeneous turbulence near a plane and is not related to the weak density variation in the mixed layer as proposed by the author in the paper referred to above. With respect to the grid frequency, the theory predicts $u_2 \propto v_2 \propto f$, but $w_2 \propto f^{\frac{3}{2}}$.

In the theory below, we emphasize the presence of an interfacial layer (region R_3) of thickness h separating the mixed layer from the non-turbulent fluid above. All observations (for example Wolanski & Brush 1975) indicate that the mixed layer is in fully developed turbulent motion with very little density variation and that the interfacial layer with its large density gradient is typified by wave motion. Wolanski & Brush found that the frequency of the disturbances in the interfacial layer was proportional to the Brunt–Väisälä frequency $(\Delta b/h)^{\frac{1}{2}}$ although numerically one order of magnitude smaller. Some authors (Crapper & Linden 1974; Hopfinger & Toly 1976) give a qualitative picture of eddies 'penetrating' the interface from the mixed layer, but a satisfactory picture of the mixing process across the interfacial layer has not yet been produced. Certainly turbulence of some kind exists in the interfacial layer and since the density gradient is strong rather than weak as in the mixed layer, it is reasonable to assume, as we do in the theory, that the turbulence in the interfacial layer is intermittent and that this intermittent, weak turbulence transfers the buoyancy in the layer. This is also the opinion of Wolanski & Brush. In the theory of this paper, the intermittency factor decreases with increasing stability, so that for the large Richardson numbers of the asymptotic theory, the layer is, for the most part, in laminar wave motion with occasional breaking waves in the interior and at the lower surface of the interface.

It is remarkable that h appears to be proportional to D and is uninfluenced by the stability. This was first noticed in cases in which two turbulent layers are separated by an interfacial layer (Long 1973; Crapper & Linden 1974) but has also been reported in experiments with only one turbulent layer as in the present paper (Wolanski & Brush 1975; Hopfinger & Toly 1976). Crapper & Linden found h/D to be constant as the Richardson number varied from 4 to 6000. Hopfinger & Toly distinguish between h_d, measured during the running of the experiment, and h_s, measured after the agitation has been stopped and the motion has come to rest. The difference between the two decreases with Richardson number, so that for large Richardson numbers $h \sim h_s$ and h_s was found to be independent of Ri. The behaviour $h = aD$ is derived below.

The theory assumes a linear density field in the interfacial layer. This is well supported by observations (Wolanski & Brush 1975) when the upper layer is either turbulent or consists of non-turbulent homogeneous fluid. When the upper fluid has a linear density gradient as in the experiments of Linden (1975) the division between the upper fluid and the fluid in the interfacial layer may become unclear (his figure 1) when the two gradients differ rather little but this should not be a serious objection to the present model.

2. Governing parameters

We begin by considering the best choice of parameters governing the phenomena in the experiment. The external quantities are the frequency of oscillation of the grid f, the stroke S, lengths $M_1, M_2, \ldots$, describing the geometry and position of the grid, the viscosity ν, and the initial density variation. We assume that the dimensions of the tank are large enough to be neglected. In a recent paper (Long 1978) the author has shown that the grid may be replaced by a virtual source of energy at a horizontal plane. The 'action' of the source is determined by a single parameter K having the dimensions of viscosity and proportional to the constant eddy viscosity in the turbulent fluid above the source. When stratification exists, the eddy viscosity will be constant in the lower portions of the mixed layer since the velocities are very high there and buoyancy effects negligible, as we shall see. The integral length scale in the region $0 < z \ll D$ is proportional to the distance z from the virtual source.

The role of the density stratification may be examined by integrating the equation

$$\partial q/\partial z = \partial \bar{b}/\partial t \tag{3}$$

first over the mixed layer and then over the interfacial layer. Let us assume a linear buoyancy field in the non-turbulent fluid with buoyancy gradient N^2. The mean buoyancy in the mixed layer is nearly constant with height and equal to

$$\bar{b}_m = b_\infty - N^2(D+h) + \Delta b, \tag{4}$$

where b_∞ is constant. We get

$$q_2 = D\frac{d\Delta b}{dt} - N^2 D\frac{d}{dt}(D+h). \tag{5}$$

In the interfacial layer (region R_3), the mean buoyancy is

$$\bar{b}_3 = \Delta b - (\Delta b/h)(z-D) + b_\infty - N^2(D+h). \tag{6}$$

Integrating (3), we get

$$q_3 = q_2 + \frac{d\Delta b}{dt}\left(\zeta - \frac{\zeta^2}{2h}\right) + \Delta b\left(\frac{\zeta^2}{2h^2}\frac{dh}{dt} + \frac{\zeta}{h}\frac{dD}{dt}\right) - N^2\zeta\left(\frac{dD}{dt} + \frac{dh}{dt}\right), \tag{7}$$

where $\zeta = z - D$. At $\zeta = h$ the buoyancy flux is zero. Using this and (5) we get

$$d[(D+\tfrac{1}{2}h)\Delta b - \tfrac{1}{2}N^2(D+h)^2]/dt = 0. \tag{8}$$

If the buoyancy field at the initial instant is the linear field $b_\infty - N^2 z$, (8) yields

$$(D+\tfrac{1}{2}h)\Delta b = \tfrac{1}{2}N^2(D+h)^2 \tag{9}$$

and N is the fundamental constant characterizing the buoyancy field. Otherwise the constant

$$V^2 = (D+\tfrac{1}{2}h)\Delta b - \tfrac{1}{2}N^2(D+h)^2 \tag{10}$$

and N may be used as fundamental constants. If the upper fluid is homogeneous, $(D+\frac{1}{2}h)\Delta b$ is constant. As indicated by experiment and by the theory of § 4, the interfacial thickness h is proportional to D, so that $U^2 = D\Delta b$ may be used as a fundamental constant when the upper fluid is homogeneous. The quantity $\hat{Ri} = U^2D^2/K^2$ has the form of a Richardson number and is useful whether or not U^2 is constant.

3. The interfacial layer (region R_3)

The energy equation at any level z is

$$0 = -\partial[\overline{\tfrac{1}{2}w(2p/\rho_0 + u^2 + v^2 + w^2)}]/\partial z - \overline{wb} - \epsilon, \tag{11}$$

where $q = -\overline{wb}$ is the buoyancy flux and ϵ is the energy dissipation. We assume that the velocities u, v and w are all of the same order of magnitude. This seems reasonable in view of our understanding of internal laminar and breaking waves. The interfacial layer is turbulent with intermittency factor I_3. Since buoyancy flux occurs only in the turbulent portions of this layer, we get, at any level in the interfacial layer R_3,

$$q_3 = -B_1 u_3 b_3 I_3, \tag{12}$$

where B_1 is a universal constant, u_3 is the r.m.s. velocity and b_3 is the r.m.s. buoyancy fluctuation in the interfacial layer.† The turbulence is certainly strongly influenced by buoyancy in this layer, so that the kinetic and the available potential energy are of the same order not only in the waves but in the turbulent patches, i.e.

$$u_3^2 = B_2 \delta_3 b_3 = B_3 \delta_3^2 \Delta b/h, \tag{13}$$

where δ_3 is of the order of the dimensions of the patches at any level in the interfacial layer. We get

$$\delta_3/h = B_3^{-\frac{1}{2}} u_3/(h\Delta b)^{\frac{1}{2}}. \tag{14}$$

Using (13), (12) becomes

$$q_3 = -\frac{B_1}{B_2} B_3^{\frac{1}{2}} \frac{u_3^3}{h} \frac{I_3 (h\Delta b)^{\frac{1}{2}}}{u_3}. \tag{15}$$

Let us now compute the dissipation. This occurs only in the turbulent patches, where $\epsilon_p = B_4 u_3^3/\delta_3$, so that

$$\epsilon_3 = B_4 I_3 u_3^3/\delta_3 = B_4 B_3^{\frac{1}{2}} I_3 u_3^3 (h\Delta b)^{\frac{1}{2}}/hu_3. \tag{16}$$

Equations (15) and (16) show that $\epsilon_3 \sim q_3$. Since these are both dissipative, it follows that they are of the order of the energy flux divergence. At the upper boundary of the interfacial layer, the kinetic energy of the waves has been so reduced by losses to potential energy and dissipation that there can no longer be wave breaking and turbulence. Thus h is the depth of penetration of the turbulence. At the height $z = D+h$, the energy flux is too weak to support turbulence, so that it has decreased to a value well below that at the bottom of the interfacial layer. Therefore, the increment in energy flux over the interfacial layer is proportional to the value at the bottom of the interfacial layer. Integrating between levels in the layer near the upper and lower surfaces, we get for the mean buoyancy flux

$$\bar{q}_3 h = -A_2 u_{30}^3, \tag{17}$$

where u_{30} is the r.m.s. velocity in the lower part of the interfacial layer. Obviously, there can be no change of order of magnitude of the vertical velocity across the level $z = D$, so that $w_2 \sim u_{30}$. Using (7), we may write

$$-\frac{A_2 w_2^3}{h} = D\frac{d\Delta b}{dt} + \frac{h}{3}\frac{d\Delta b}{dt} + \frac{\Delta b}{6}\frac{dh}{dt} + \tfrac{1}{2}\Delta b\frac{dD}{dt} - N^2(D+\tfrac{1}{2}h)\frac{d}{dt}(D+h), \tag{18}$$

$$\delta_2/h = B_3^{-\frac{1}{2}} w_2/(h\Delta b)^{\frac{1}{2}}, \tag{19}$$

$$B_3^{\frac{1}{2}} \frac{B_1}{B_2} I_2 w_2^2 \left(\frac{\Delta b}{h}\right)^{\frac{1}{2}} = -D\frac{d\Delta b}{dt} + N^2 D \frac{d}{dt}(D+h), \tag{20}$$

where I_2 and δ_2 are evaluated at a level just above $z = D$.

† We use $B_1, B_2, \ldots$ to denote universal constants and $A_1, A_2, \ldots$ to denote 'constants' which vary with the stability of the upper, non-turbulent fluid. Rough estimates of the important constants of the paper are given in the appendix.

It is of interest to compute the energy flux in the lower part of the interfacial layer. In the eddies, the contribution is of order $I_2 w_2^3$. This is small compared with the total flux w_2^3, so that the transfer is due primarily to the waves and is of order

$$w_2 p_2 \sim w_2^2 c,$$

where c is the speed of the energy-containing waves of length λ, i.e. $c \sim \lambda(\Delta b/h)^{\frac{1}{2}}$. Since $w_2^2 c$ is of the order of w_2^3, $c \sim w_2$, or

$$\lambda \sim w_2(h/\Delta b)^{\frac{1}{2}}. \tag{21}$$

According to (19), λ is of order of the length and amplitude of the breaking waves. It appears that pressures in the eddies in region R_2 of frequency w_2/λ of the order of the natural frequency $(\Delta b/h)^{\frac{1}{2}}$ are generating the breaking waves by resonance. The layer R_2 has a thickness of the order of the amplitude δ_2 of the waves on the lower surface of the interfacial layer.

4. Turbulence in the mixed layer and final results

So far we have simply assumed turbulence below the interfacial layer without considering its properties in detail and the relation of these to the flux q_2 or to the entrainment velocity u_e. Let us consider the effect of buoyancy in the layer R_2, where the magnitude of w is w_2. Here $q_2 \sim w_2 b_2 \sim w_2^3/h$. Thus $b_2 h \sim w_2^3$, so that the ratio of kinetic to available potential energy of the eddies, which is at least as large as $w_2^2/b_2\delta_2$, is

$$\frac{T}{V} \geqslant \frac{w_2^2}{b_2\delta_2} \sim \frac{h}{\delta_2} \sim \frac{(h\Delta b)^{\frac{1}{2}}}{w_2}. \tag{22}$$

This is very large for the highly stable conditions of this paper, so that the buoyancy variation is unimportant dynamically in R_2. This will certainly also be true in the rest of the mixed layer, so that in the whole mixed layer mean quantities are determined by factors independent of the buoyancy and depend only on K, D and distance z. We are concerned, particularly, with finding w_2 in terms of K and D and the distance $\zeta = D - z$ from the lower surface of the interfacial layer. We do this by using the results of Hunt & Graham (1978) for the distortion of homogeneous turbulence by the presence of a rigid wall. The problem in the present paper is somewhat different because the turbulence in the mixed layer far from the interface is not homogeneous in the vertical direction but this does not have basic importance.

Let us first assume that the surface at $z = D$ is rigid. The presence of the surface then requires that the vertical component of the velocity tends to zero as $\zeta \to 0$. The pressure forces will cause energy to be transferred to the horizontal components. Since $w_2 \to 0$ as $\zeta \to 0$ and since the total kinetic energy is conserved near the surface (Hunt & Graham 1978), u_2^2 and v_2^2 will increase to $\frac{3}{2}$ of their values at $z = D$ in the absence of the surface. Thus $u_2 \sim v_2 \sim u_1 \sim K/D$, but w_2 will be much smaller if ζ is small. At small ζ eddies of length much less than ζ will not feel the distorting effect of the surface and will be isotropic. Eddies of length much greater than ζ will feel the surface very strongly and will be strongly flattened, while eddies of length of order ζ will feel the surface but will remain quasi-isotropic. From the equation of continuity, the large flattened eddies of horizontal dimensions D yield $w_{2f} \sim u_1 \zeta/D \sim K\zeta$. The quasi-isotropic eddies will have a spectrum function

$$E_i(k) \sim \epsilon_2^{\frac{2}{3}} k^{-\frac{5}{3}}, \quad k \sim \zeta^{-1},$$

where k is the wavenumber and ϵ_2 is the dissipation function near $\zeta = 0$, so that the contribution is $w_{2i} \sim \epsilon_2^{\frac{1}{3}} \zeta^{\frac{1}{3}}$. This is much larger than w_{2f}, so that $w_2 \sim \epsilon_2^{\frac{1}{3}} \zeta^{\frac{1}{3}}$. For large but finite Richardson numbers, the surface will not be flat but will be agitated by disturbances of amplitude δ_2 and we infer that $w_2 \sim \epsilon_2^{\frac{1}{3}} \delta_2^{\frac{1}{3}}$ in R_2. Of course, $\epsilon_2 \sim K^3/D^4$, so that $w_2 \sim K\delta_2^{\frac{1}{3}}/D^{\frac{4}{3}}$, or

$$w_2^3 D^4/\delta_2 K^3 = B_6. \tag{23}$$

Using (18)–(20) and (23), we may write

$$\frac{w_2 D}{K} = B_6^{\frac{1}{2}} B_3^{-\frac{1}{4}} \frac{K^{\frac{1}{2}}}{D}\left(\frac{h}{\Delta b}\right)^{\frac{1}{4}}, \tag{24}$$

$$-\frac{A_2 B_6^{\frac{3}{2}} B_3^{-\frac{3}{4}} K^{\frac{9}{2}}}{h^{\frac{1}{4}} D^6 (\Delta b)^{\frac{3}{4}}} = D\frac{d\Delta b}{dt} + \frac{h}{3}\frac{d\Delta b}{dt} + \frac{\Delta b}{6}\frac{dh}{dt} + \tfrac{1}{2}\Delta b \frac{dD}{dt} - N^2(D + \tfrac{1}{2}h)\frac{d}{dt}(D+h), \tag{25}$$

$$\frac{\delta_2}{D} = B_6^{\frac{1}{2}} B_3^{-\frac{3}{4}} \frac{K^{\frac{3}{2}}}{D^3}\left(\frac{h}{\Delta b}\right)^{\frac{3}{4}}, \tag{26}$$

$$-\frac{B_1 B_6}{B_2} I_2 \frac{K^3}{D^4} = D\frac{d\Delta b}{dt} - N^2 D \frac{d}{dt}(D+h). \tag{27}$$

We now find a relationship between h and D. The dissipation function in the patches in R_3 is of order u_3^3/δ_3. This is independent of $\hat{Ri}$ in the lower portions of the layer, as we see in (23), and should remain independent of $\hat{Ri}$ in the whole layer although it will certainly vary with ζ/D and, perhaps, parameters expressing initial conditions and the stability of the upper layer. If we suppress the dependence on these parameters, we may write

$$\frac{u_3^3}{\delta_3} = \frac{K^3}{D^4}\phi\left(\frac{\zeta}{D}\right). \tag{28}$$

Using (14), we obtain

$$u_3^3 = \left(\frac{h}{\Delta b}\right)^{\frac{3}{4}} \frac{K^{\frac{9}{2}}}{D^6}\phi_1\left(\frac{\zeta}{D}\right). \tag{29}$$

We may obtain another expression for u_3^3 by integrating the energy equation over the interfacial layer. We have already seen that $\epsilon_3 \sim q_3$ and that the energy flux is proportional to u_3^3 in this layer, so that

$$\partial u_3^3/\partial\zeta = B_7 q_3. \tag{30}$$

Using (7) and integrating, we get

$$u_3^3 = w_2^3 + B_7\left[q_2\zeta + \frac{d\Delta b}{dt}\left(\frac{\zeta^2}{2} - \frac{\zeta^3}{6h}\right) + \Delta b\left(\frac{\zeta^3}{6h^2}\frac{dh}{dt} + \frac{\zeta^2}{2h}\frac{dD}{dt}\right) - N^2\frac{\zeta^2}{2}\frac{d}{dt}(D+h)\right]. \tag{31}$$

Comparison of (29) and (31) shows that

$$\phi_1\left(\frac{\zeta}{D}\right) = B_6^{\frac{3}{2}} B_3^{-\frac{3}{4}} - A_3\frac{\zeta}{D} + A_4\frac{\zeta^2}{D^2} + A_5\frac{\zeta^3}{D^3}. \tag{32}$$

Equating coefficients, we get

$$w_2^3 = B_6^{\frac{3}{2}} B_3^{-\frac{3}{4}}\left(\frac{h}{\Delta b}\right)^{\frac{3}{4}}\frac{K^{\frac{9}{2}}}{D^6} \tag{33}$$

in agreement with (24) and, using (5),

$$D\frac{d\Delta b}{dt} - N^2 D\frac{d}{dt}(D+h) = -\frac{A_3}{B_7}\left(\frac{h}{\Delta b}\right)^{\frac{3}{4}}\frac{K^{\frac{9}{2}}}{D^7}, \tag{34}$$

$$\frac{D}{2}\frac{d\Delta b}{dt} + \frac{\Delta b}{2h}D\frac{dD}{dt} - N^2\frac{D}{2}\frac{d}{dt}(D+h) = \frac{A_4}{B_7}\left(\frac{h}{\Delta b}\right)^{\frac{3}{4}}\frac{K^{\frac{9}{2}}}{D^7}, \tag{35}$$

$$-\frac{D^2}{6h}\frac{d\Delta b}{dt} + \frac{D^2}{6h^2}\Delta b\frac{dh}{dt} = \frac{A_5}{B_7}\left(\frac{h}{\Delta b}\right)^{\frac{3}{4}}\frac{K^{\frac{9}{2}}}{D^7}. \tag{36}$$

Equations (10), (25) and (34)–(36) together with initial conditions determine the problem. Let us find solutions for two important cases.

(*a*) *Homogeneous upper fluid* ($N^2 = 0$). When the upper fluid is homogeneous, (10) becomes

$$(D+\tfrac{1}{2}h)\,\Delta b = V_0^2 = \text{constant}. \tag{37}$$

We then find that all equations are invariant under the following transformation:

$$\left.\begin{aligned} t = a_1 t', \quad D = a_1^{\frac{2}{11}} K^{\frac{9}{11}} V_0^{-\frac{7}{11}} D', \quad h = a_1^{\frac{2}{11}} K^{\frac{9}{11}} V_0^{-\frac{7}{11}} h', \\ \Delta b = V_0^{\frac{29}{11}} a_1^{-\frac{2}{11}} K^{-\frac{9}{11}} \Delta b', \end{aligned}\right\} \tag{38}$$

where a_1 is an arbitrary constant. The general solution for D is

$$D/(Kt)^{\frac{1}{2}} = f(V_0 t^{\frac{1}{2}}/K^{\frac{1}{2}}). \tag{39}$$

Because a_1 is arbitrary, the solution has the form

$$D = B_8 V_0^{-\frac{7}{11}} K^{\frac{9}{11}} t^{\frac{2}{11}}, \quad h = aD, \quad \Delta b = 2V_0^2/(2+a)\,D, \tag{40}$$

where B_8 and a are universal constants. The solution involves a virtual origin of time $t = 0$ when D is zero and Δb is infinite but the product $D\Delta b$ is finite. We have remarked that $h = aD$ is observed in experiments with $a \simeq 0{\cdot}25$ (Crapper & Linden 1974). Equation (40) corresponds to an entrainment velocity

$$u_e D/K = \alpha_2 \hat{R}i^{-\frac{7}{4}}, \quad \alpha_2 = 6A_2 B_6^{\frac{3}{2}}/a^{\frac{1}{4}} B_3^{\frac{3}{4}}(3+a), \tag{41}$$

where α_2 is a universal constant. As shown in experiments (Hopfinger & Toly 1976) and in a theory by the author (1978), u_1 (or K) is proportional to the grid frequency f, so that (41) leads to

$$u_e/fS \sim Ri^{*-\frac{7}{4}}, \quad Ri^* = D\Delta b/f^2S^2, \tag{42}$$

compared with the $Ri^{*-\frac{3}{2}}$ law suggested by experimenters. Inspection of their data, however, reveals little reason to choose one law in preference to the other. Other results are

$$w_2 D/K = \alpha_1 \hat{R}i^{-\frac{1}{4}}, \quad \alpha_1 = B_6^{\frac{1}{2}} a^{\frac{1}{4}}/B_3^{\frac{1}{4}}, \tag{43}$$

$$\delta_2/D = \alpha_3 \hat{R}i^{-\frac{3}{4}}, \quad \alpha_3 = B_6^{\frac{1}{2}} a^{\frac{3}{4}}/B_3^{\frac{3}{4}}, \tag{44}$$

$$I_2 = \alpha_4 \hat{R}i^{-\frac{3}{4}}, \quad \alpha_4 = 6A_2 B_6^{\frac{1}{2}} B_2/B_1 a^{\frac{1}{4}} B_3^{\frac{3}{4}}(3+a), \tag{45}$$

where the α_i are universal constants.

(*b*) *Linear gradient in upper layer* ($N^2 \neq 0$). In the case of a linear gradient in the upper layer, (10) yields $V^2 = 0$ if we trace back to $t = 0$, $\Delta b = 0$, $D = 0$. We now find that all equations are invariant under the following transformation:

$$\left.\begin{aligned} t = a_1 t', \quad D = a_1^{\frac{1}{9}} K^{\frac{1}{2}} N^{-\frac{7}{18}} D', \quad h = a_1^{\frac{1}{9}} K^{\frac{1}{2}} N^{-\frac{7}{18}} h', \\ \Delta b = N^{\frac{29}{18}} a_1^{\frac{1}{9}} K^{\frac{1}{2}} \Delta b', \end{aligned}\right\} \tag{46}$$

where a_1 is arbitrary. The general solution for D is

$$D/(Kt)^{\frac{1}{2}} = f(Nt). \tag{47}$$

Because a_1 is arbitrary, the solution for the initial condition $D = 0$ at $t = 0$ is

$$D = B_9 K^{\frac{1}{2}} N^{-\frac{7}{18}} t^{\frac{1}{9}}, \quad h = aD, \quad R = N^2 h/\Delta b = (2+a)a/(1+a)^2, \tag{48}$$

where B_9 and a are universal constants. It is remarkable that the ratio of the buoyancy gradients in the upper fluid and the interfacial layer is a universal constant. If $a = 0{\cdot}25$, this ratio is $R = 0{\cdot}36$. The Richardson number behaviour for the linear case is the same as in (41) and (43)–(45).

In both case (a) and case (b) the buoyancy fluctuations are given by

$$b_1/\Delta b \sim \hat{Ri}^{-\frac{7}{4}}, \quad b_2/\Delta b \sim \hat{Ri}^{-\frac{3}{2}}, \tag{49}$$

which verifies that they have no dynamic effect in the mixed layer.

5. Summary and conclusions

It is now possible to form a reasonable description of the state of affairs in the experiment. The oscillating grid is nearly equivalent to a source of energy on a plane ($z = 0$) as discussed by the author (1978). A single parameter K characterizes the source, where K has dimensions L^2T^{-1}. If the fluid is homogeneous the source, started at $t = 0$, causes the development of a turbulent layer of depth $D \sim (Kt)^{\frac{1}{2}}$ separated by a front from the non-turbulent fluid above. Thus the front propagates at a speed proportional to $t^{-\frac{1}{2}}$. After the front has moved far away, conditions ultimately reach a steady state with energy supplied by the source. In any layer there is a balance between the energy entering at the lower plane less the energy leaving through the upper plane and the energy dissipation. The r.m.s. velocity components are proportional to each other with universal constants of proportionality. The horizontal component, for example, is given by

$$uz/K = \chi_1(K/\nu) \tag{50}$$

and the integral length scale by

$$l/z = \chi_2(K/\nu), \tag{51}$$

where ν is the molecular viscosity. The energy dissipation ϵ decreases rapidly with distance from the grid: as K^3/z^4.

If the fluid is initially stratified, the energy source creates a mixed layer of thickness D, which increases much more slowly with time, separated from the non-turbulent fluid above by an interfacial layer of thickness h. Although h is small compared with D, it remains proportional to D. The mixed layer has a very weak buoyancy gradient, so that dissipation nearly balances the energy flux divergence, and the loss of kinetic energy to potential energy is small. This is reflected in the fact that the kinetic energy u^2 is much greater than the available potential energy lb, where l is the eddy scale and b is the perturbation buoyancy. The eddy length l is proportional to distance from the source, becoming of order D in the centre of the layer.

The r.m.s. velocity u varies as K/z in the lower part of the mixed layer, decreasing with z but maintaining a proportionality to K. In the experiment, if viscosity is negligible, K is proportional to the frequency of the grid f. As we approach the interface the eddies become quasi-horizontal, the important vertical component decreases to a value w_2 and quasi-isotropic eddies of size δ_2 and velocity w_2 disturb the interfacial surface, creating waves of length δ_2 and fluid velocity w_2. If the stability is infinitely great, i.e. as $\hat{Ri} \to \infty$, the r.m.s. velocity w_2 tends to zero at the interface. For finite but large values of $\hat{Ri}$, we have $w_2 \sim (K/D)\hat{Ri}^{-\frac{1}{4}}$ and $\delta_2 \sim D\hat{Ri}^{-\frac{3}{4}}$. This dimension is also the dimension of the layer over which the interface moves up and down. This length was measured by Hopfinger & Toly as proportional to Ri^{*-1}, where Ri^* is proportional to $\hat{Ri}$, which is not far from the $\hat{Ri}^{-\frac{3}{4}}$ behaviour predicted by the theory.

The interfacial layer is very stable and has intermittent turbulence with intermittency factor $I_2 \sim \hat{Ri}^{-\frac{3}{4}}$. The turbulence in patches of dimension δ_2 is caused by breaking waves of length and amplitude δ_2. These are energized by resonance with the pressure fluctuations in the quasi-isotropic eddies of length δ_2 and frequency w_2/δ_2 equal to the natural frequency $(\Delta b/h)^{\frac{1}{2}}$ acting on the lower interfacial surface.

Finally, the buoyancy flux varies in both the mixed layer and the interfacial layer. It is a maximum q_2 at the level $z = D$, where it is of order w_2^3/D and thus determined by the r.m.s. vertical velocity in the mixed layer close to the interface. $q_2 \sim u_e \Delta b$, where u_e is the entrainment velocity or rate of increase of D and is proportional to $\hat{Ri}^{-\frac{7}{4}}$. This is close to measurements, but differs slightly from the $Ri^{*-\frac{3}{2}}$ law proposed by the experimenters. The entrainment law leads to a variation of D as $t^{\frac{2}{11}}$ for a homogeneous upper fluid compared with $D \propto t^{\frac{1}{6}}$ when the upper fluid has a linear density gradient. The $t^{\frac{1}{6}}$ law reveals a slower rate of increase for the linear case as expected but it is somewhat slower than that measured by Linden (1975).

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Appendix

We evaluate the important constants in the paper for a homogeneous upper fluid as best we can, using reasonable values when possible and data obtained by Hopfinger & Toly (1976). Reasonable estimates are $B_1 = 0{\cdot}4$, $B_2 = 0{\cdot}4$, $B_3 = 1$ and $B_4 = 1$. The constant A_2 in (17) is more difficult. If we estimate the energy flux as $3u_3^3$ and estimate that u_3^3 drops off to $0{\cdot}1w_2^3$ at the top of the interfacial layer, we get $A_2 = 1{\cdot}35$. The constant B_6 is very uncertain and we shall estimate α_2 instead in (41). The data of Hopfinger & Toly for a representative experiment yield

$$\left.\begin{aligned} &a = 0{\cdot}25, \quad l\Delta b/u^2 = 22, \quad u_e/u = 0{\cdot}02, \quad l = 3{\cdot}44\,\text{cm}, \quad D = 20{\cdot}5\,\text{cm}, \\ &u = 1{\cdot}92\,\text{cm}\,\text{s}^{-1}, \quad K = 6{\cdot}6\,\text{cm}^2\,\text{s}^{-1}, \quad \hat{Ri} = 4662, \end{aligned}\right\} \tag{A 1}$$

so that $\alpha_2 = 3{\cdot}1 \times 10^5$. This yields $B_6 = 1978$, so that $(w_2^3/\delta_2)(u^3/l)^{-1}$ is approximately $1{\cdot}6$. We may also compute $B_7 = 0{\cdot}67$. The α_i in (41) and (43)–(45) are

$$\alpha_1 = 31, \quad \alpha_2 = 3{\cdot}1 \times 10^5, \quad \alpha_3 = 15{\cdot}7, \quad \alpha_4 = 157. \tag{A 2}$$

For the conditions of the experiment these yield

$$w_2 = 1{\cdot}21\,\text{cm}\,\text{s}^{-1}, \quad \delta_2 = 0{\cdot}57\,\text{cm}, \quad I_2 = 0{\cdot}28. \tag{A 3}$$

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